Markov Chains – part 1

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Let us consider the following problem:

Two players, A and B, are playing chess, with A starting. Each move can be either strong or weak. The first player to respond with a strong move to the opponent’s weak move wins the game. Player A makes a strong move with probability $x$ and a weak move with probability $1 - x$. Similarly, player B has probabilities $y$ and $1 - y$. We assume that $0 < x < 1$ and $0 < y < 1$. For which values of $x$ and $y$ are the chances of winning for A and B equal?

Solution. Note that as long as nobody has won, the situation depends only on the last move. There are four possibilities: a strong/weak move by player A/B. Let us denote them by $A+$, $A-$, $B+$, and $B-$. The moment when player A starts can be considered separately, but there is no need for that because it is equivalent to the situation $B+$. So, we can take $B+$ as the starting point. Additionally, we include states $A$ (indicating that A has won) and $B$ (indicating that B has won). The diagram shows all six situations, with assigned numbers from 1 to 6, and connected by arrows representing transition probabilities.

We call such an object a finite Markov chain. It is described by a set of states $S_1, S_2, \ldots, S_6$ and transition probabilities $p_{ij}$ for moving from state $S_i$ to state $S_j$ in one step for all $i, j \in 1, 2, \ldots, n$. This means that when in state $S_i$, the chain will transition to state $S_j$ with probability $p_{ij}$, and so on. For each $i$, the equation $p_{i1} + p_{i2} + \ldots + p_{in} = 1$ holds. States $S_7$ from which there are no outgoing transitions ($p_{ij} = 0$) are called absorbing.

We are interested in the probability of player A winning, which is the probability that the process ends in state $S_6$, given that it started in state $S_1$. To approach this problem, let $q_6$ denote the probability that the process ends in state $S_6$, given that it started in state $S_1$. Note that $q_6 = 1$ and $q_5 = 0$. From state $S_1$, we can transition to $S_2$ (with probability $x$) or $S_3$ (with probability $1 - x$). This implies that $q_1 = q_4 + (1 - x)q_3$. Similarly, we obtain the equations:

$$q_2 = yq_1 + (1 - y)q_4, \quad q_3 = (1 - y)q_4, \quad q_4 = (1 - x)q_3 + x.$$

By solving this system of four equations with variables $q_1, q_2, q_3, q_4$, we obtain:

$$\frac{1}{x} = q_1 = \frac{x(1 - y)}{(1 - xy)(x + y - xy)} \iff x - y = xy(1 - x)(1 - y),$$

which means that for $x$ and $y$ satisfying the last equality, both players have an equal chance of winning. Note that it must hold that $x > y$, since the right-hand side of the last equality is positive.

Problems

1. The kitten wanders between the house, kindergarten, garden, field, and forest. It starts in the garden and always chooses one of the remaining four places. It always plays in the house and then goes to the field or back to the garden. After a walk in the forest, the kitten always goes either to the garden or to the kindergarten. If the kitten reaches the kindergarten, it never leaves from there (children, you know...). On the field, however, the kitten catches a mouse and ends its wanderings. Calculate the probability of ending up in the field. (We assume that the kitten’s choices are random and equally likely.)

2. A student wants to buy their favorite energy drink, which costs 5 złotych. Unfortunately, the student has only 2 złote. So, they decide to go to a casino where, with a probability of $p$, they can win 3 złote for each złoty bet, or lose the bet with a probability of $1 - p$. The student stops playing when they have enough money to buy the drink or when they run out of money. Depending on the value of $p$, determine the chances of the student achieving their goal. (Note: The author of this column does not endorse energy drinks or gambling in any way.)

3. Hansel and Gretel toss a fair coin. If the sequence $HHT$ (where $H$ denotes heads and $T$ denotes tails) appears in three consecutive tosses, Hansel wins the game. If the sequence $HHT$ appears, Gretel wins. Determine the probability of Hansel winning.

4. Seven children are standing in a circle, playing with a ball. Each child who currently has the ball throws it to the child standing immediately to their left (with probability $p < \frac{1}{2}$) or to the child standing immediately to their right (also with probability $p$), or takes the ball and goes back home (with probability $1 - 2p$). Depending on the value of $p$, calculate the probability that the same child who brought the ball home will return with it.